

Existence of periodic wave of a BBM equation with delayed convection and weak diffusion

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Abstract This paper focus on the existence and uniqueness of periodic waves for a BBM equation with local strong generic delay convection and weak diffusion. By analyzing the corresponding Hamiltonian system, we aim to obtain the existence of periodic orbit by constructing a locally invariant manifold according to geometric singular perturbation theory. Chebyshev criteria is applied to investigate the ratio of Abelian integrals. We prove the existence and uniqueness of periodic wave solution with sufficiently small perturbation parameter. Moreover, the upper and lower bounds of the limiting wave speed are given.

Keywords delayed BBM equation · geometric singular perturbation theory · periodic wave · Abelian integrals

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1 Introduction

Traveling wave solutions plays an important role in many mathematically modelled phenomena. It can be applied in comparison principles and characterize the long-term behaviour in numerous situations in conformance with

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their natural occurrence. Up to now, there are lots of nonlinear models, such as, Korteweg-de Vries equation [1], Green-Naghdi [3], Burgers equation [4], Kadomtsev-Petviashvili equation [2], Camassa-Holm equation [5]. A well known nonlinear, dispersive equation is Benjamin-Bona-Mahony (BBM) equation [6], which given by

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad (1)$$

which is deduced basing on model with nonlinear dispersion under the same assumption as the KdV equation, describing unidirectional propagation of weakly long surface waves. According to the widely application in surface waves of long wavelengths, hydromagnetic waves in cold plasma, acoustic gravity waves in compressible fluids etc.[7], BBM equation was modified and generalized into many forms. Moreover, since BBM equation has an advantages over KdV equation on investigating the long waves, researchers in many fields had been attracted to investigate it from different points. For example, Micu [8] proved the BBM equation is approximately controllable but not spectrally controllable and proved a finite controllability result. New soliton solutions of modified BBM equation were constructed by generalized Kudryashov method [9]. By using the theory of bifurcations of dynamical systems, Zhao [10] proved there exist some smooth and non-smooth periodic wave solutions, solitary wave solutions for some generalized BBM equations. By inverse scattering transformation, Biswas [12] obtained the solitary wave solutions to the BBM equation with dual power law nonlinearity. By Lie-group method, Singh [11] studied the symmetries of the BBM equation with variable coefficients.

Particularly, Wazwaz [13] studied the traveling wave solutions on following generalized forms of the BBM equation

$$(u^m)_t + (u^n)_x + (u^l)_{xxx} = 0 \quad (2)$$

and found its compaction of dispersive structures. After that, Chen et al. [14] considered the BBM equation (2) with $m = 2, n = 3$ and $l = 1$

$$(u^2)_t + (u^3)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxx}) = 0, \quad (3)$$

which including weak backward diffusion and dissipation effects. By using geometric singular perturbation theory, they established the existence of periodic waves and solitary waves. Picard-Fuchs equation was introduced to studied the properties on the ratios of Abelian integrals. Zhu [15] gave the persistence of two solitary waves with particular nonzero limits for (3) when the integral constant is taken into consideration. Zhang [16] presented a portion of the solitary wave solutions for the equation (3) persist under small perturbations and the wave speed selection principle. Sun and Yu [17] investigated the BBM equation (2) for $m = 3, n = 4$ and $l = 1$ with two different perturbations

$$(u^3)_t + (u^4)_x + u_{xxx} + \varepsilon P_i = 0, \quad i = 1, 2, \quad (4)$$

with $P_1 = u_{xx} + u_{xxx}$, $P_2 = ((\alpha_0 + \alpha_1 u + \alpha_2 u^2)u_x)_x$, where α_0, α_1 and α_2 are bounded parameters. The authors focused on the existence of periodic waves

on a locally invariant manifold and found the number of zeros of some linear combinations of Abelian integrals. The analysis is mainly based on Chebyshev criteria and asymptotic expansions of Abelian integrals near the solitary and singularity. For the equation (4) with P_2 , they gave the asymptotic expansions of the Melnikov functions at two endpoints and proved the upper bounded number of periodic waves is two. The sufficient conditions for the coexistence of periodic waves and solitary wave were given. Guo and Zhao [18] studied the BBM equation (2) for $m = 3, n = 5$ and $l = 1$ with weak backward diffusion and dissipation effects given by

$$(u^3)_t + (u^5)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0. \quad (5)$$

They proved that periodic wave solutions persist for sufficiently small perturbation parameter. By Picard-Fuchs equations and linear programming, the property of ratio of Abelian integrals were given. More recently, Dai and Wei [19] investigated to the existence and uniqueness of periodic waves for a perturbed sextic generalized BBM equation with weak backward diffusion and dissipation effects

$$(u^5)_t + (u^6)_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0. \quad (6)$$

The associated ODE of (6) with $\varepsilon = 0$ has a more degenerate singularity, that is, the nilpotent saddle with order 2.

There are some other excellent works on this topic, for instance, Ogawa [20] studied the traveling wave solutions to a perturbed KdV equation. Yan [21] investigated the existence of solitary waves and periodic waves to a perturbed generalized KdV equation. Chen [22] established the existence of kink waves and periodic waves for a perturbed defocusing mKdV equation. Sun [23] proved the coexistence of the solitary wave and one periodic wave for a convecting shallow water fluid model. Additionally, Du and his cooperators considered the existence of traveling waves solutions for a series of physical models containing convolution kernels by using the geometric singular perturbation analysis, see [24–29] in details. Therefore, geometric singular perturbation theory is an effective tools to deal with the singular problems on existence of traveling wave solution for perturbed models.

More recently, Wang [30] considered the following generalized BBM equation with weak generic kernel delay

$$u_t + ((f * u)u^{n-1})_x - u_{xxt} + \tau u_{xxxx} = 0,$$

with $n \in \mathbb{Z}^+$, f is weak generic kernel. They proved the existence of solitary wave solution under some parametric conditions. However, the existence of periodic wave solution for the delayed BBM equation didn't been considered in their manuscript. However, one may ask that, what is the dynamic behavior on the periodic wave solutions for delayed BBM equation? What's more, when BBM equation contains a strong generic kernel delay in the convection term and weak backward diffusion, does the periodic wave vanish or persist? To answer these questions, we consider a generalized BBM equation contains a

strong generic kernel delay in the convection term and weak backward diffusion given by

$$(u^2)_t + ((f * u)u^2)_x + u_{xxx} + \tau u_{xx} = 0 \quad (7)$$

with $\tau > 0$ is a perturbation parameter, $f(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}$ is the kernel function. $f * u$ is a convolution in the spatial-temporal variable and $((f * u)u^2)_x$ means a strong generic kernel delay in the convection term. τu_{xx} represents weak backward diffusion. The convolution $f * u$ is defined by

$$(f * u)(x, t) = \int_{-\infty}^t f(t-s)u(x, s)ds.$$

The kernel function f satisfies the normalization condition

$$f : [0, +\infty) \rightarrow [0, +\infty), \quad \int_0^\infty f(t)dt = 1, \quad tf(t) \in L^1((0, \infty), R).$$

Another typical kernel is usually considered in the literatures in the form $f(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$, which is called the weak generic delay kernel. The weak generic delay kernel means the idea that the further one is influenced by the past and the influence is decreasing to zero as time progresses. The local strong generic delay kernel is usually regarded as the smoothed version of the discrete delay $f(t) = \delta(t - \tau)$. According to kernel achieves its unique maximum at $t = \tau$, the strong kernel implies that some particular time in the past is more important than any others. In this paper, we will consider the strong kernel in Eq. (7), since the maximum of the perturbation may be more important in propagation of shallow water waves. However, the existence of traveling wave for Eq. (7) is a singular problem, therefore, we use geometric singular perturbation theory to reduce the singular perturbed model into regular system. Some relatively new theory of weak Hilberts 16th problem and qualitative theory of ordinary differential equations are employed to prove the presented results.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary results including geometric singular perturbation theory and system reduction. In Section 3, we investigate the existence of periodic wave solution for system (7) and give the proof of our main results. Chebyshev criteria is applied to investigate the ratio of Abelian integrals, which seem to be more effective than Picard-Fuchs equation method used in [14, 18]. It is a simplify numerical simulations on the the ratio of Abelian integrals and periodic wave solution in Section 4.

2 Preliminaries

Geometric singular perturbation theory is an effective tool to prove the traveling wave persists for the singular perturbed systems containing sufficiently small parameters, so the theory is introduced firstly, according to [31, 32].

Consider the system

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \quad (8)$$

where $t = \frac{d}{dt}$, $0 < \varepsilon \ll 1$ is a real and small parameter, $x = (x_1, x_2, \dots, x_k)^T \in R^k$, $y = (y_1, y_2, \dots, y_l)^T \in R^l$. The functions f and g are both C^∞ on a set $U \times I$, where $U \subset R^{k+l}$ is open and I is an open interval containing 0.

With a change of time scaling $\tau = \varepsilon t$, system (8) can be written as

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = g(x, y, \varepsilon), \end{cases} \quad (9)$$

where $\dot{} = \frac{d}{d\tau}$. The time scale τ is slow and t is fast. When $\varepsilon \neq 0$, systems (8) and (9) are equivalent. Thus system (8) is called *the fast system*, while system (9) is called *the slow system*. Letting $\varepsilon \rightarrow 0$ in (8), it obtains the *layer system*

$$\begin{cases} x'(t) = f(x, y, 0), \\ y'(t) = 0. \end{cases} \quad (10)$$

Here, x is called the fast variable, whereas y is called the slow variable. Let $\varepsilon \rightarrow 0$ in (9), the limit system is given by

$$\begin{cases} f(x, y, 0) = 0, \\ \dot{y} = g(x, y, 0). \end{cases} \quad (11)$$

It only makes sense if $f(x, y, 0) = 0$. Assume that there is a given an l -dimensional manifold M_0 which is contained in the set $\{f(x, y, 0) = 0\}$.

Definition 1 The manifold M_0 is *normally hyperbolic* if the linearization of layer system (10) at each point in M_0 has exactly l eigenvalues on the imaginary axis.

Definition 2 A set M is locally invariant under the flow from (8) if it has neighborhood V so that no trajectory can leave M without also leaving V . In other words, it is locally invariant if for all $x \in M$, $x \cdot [0, t] \subset V$ implies that $x \cdot [0, t] \subset M$, similarly with $[0, t]$ replaced by $[t, 0]$, when $t < 0$, where the notation $x \cdot t$ is used to denote the application of a flow after time t to the initial condition x .

Assume that there is a C^∞ function $h^0(y)$, for $y \in K$, with K being a compact domain in R^l , such that $M_0 = \{(x, y) : x = h^0(y)\}$. Consequently, the following lemmas on geometric theory of singular perturbation are established.

Lemma 1 For $\varepsilon > 0$ is sufficiently small, there exists a manifold M_ε lying within $O(\varepsilon)$ of M_0 . M_ε is diffeomorphic to M_0 and locally invariant under the flow of (8), and C^r in x, y and ε , for any $0 < r < +\infty$.

Lemma 2 For $\varepsilon > 0$ is sufficiently small, there exists a function $x = h^\varepsilon(y)$ defining on K , such that the graph

$$M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\},$$

is locally invariant under (8). Moreover, $h^\varepsilon(y)$ is C^r , for any $0 < r < +\infty$, jointly in y and ε . M_ε possesses locally invariant stable and unstable manifold $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ lying within $O(\varepsilon)$ and being C^r diffeomorphic to the stable and unstable manifold $W^s(M_0)$ and $W^u(M_0)$ of the critical manifold M_0 .

3 System reduction and perturbation analysis

For equation (7) with a local strong delay kernel, making a traveling wave transformation $u(x, t) = \tilde{u}(x + ct) = \tilde{u}(\xi)$, where $c > 0$ is the wave speed, it is changed to the following traveling wave system

$$c(\tilde{u}^2)' + (\omega \tilde{u}^2)' + \tilde{u}''' + \tau \tilde{u}'' = 0,$$

where $'$ is derivative respect to ξ , $\omega := (f * \tilde{u})(\xi) = \int_0^{+\infty} \frac{t}{\tau^2} e^{-\frac{t}{\tau}} \tilde{u}(\xi - ct) dt$. Integrating it once and neglecting the integral constant, then it has

$$c\tilde{u}^2 + \omega \tilde{u}^2 + \tilde{u}'' + \tau \tilde{u}' = 0. \quad (12)$$

Making variable transformations $\tilde{u} = c\phi$, $\xi = \frac{1}{c}z$, Eq. (12) can be simplified to

$$\phi^2 + \eta \phi^2 + \frac{\tau}{c} \frac{d\phi}{dz} + \frac{d^2\phi}{dz^2} = 0, \quad (13)$$

where

$$\eta = \int_0^{+\infty} \frac{s}{\tau^2} e^{-\frac{s}{\tau}} \phi \left(\frac{1}{c}z - cs \right) ds.$$

By direct computation, the derivatives is given by

$$\frac{d\eta}{dz} = \frac{1}{c^2}(\zeta - \eta), \quad \frac{d\zeta}{dz} = \frac{1}{c^2}(\phi - \zeta), \quad (14)$$

with

$$\zeta = \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{s}{\tau}} \phi \left(\frac{1}{c}z - cs \right) ds.$$

Introducing a new variable $\varphi := \frac{d\phi}{dz}$, combining (14) with (13), it obtains the following four-dimensional system

$$\begin{cases} \frac{d\phi}{dz} = \varphi, \\ \frac{d\varphi}{dz} = -\phi^2 - \eta \phi^2 - \frac{\tau}{c} \varphi, \\ \tau \frac{d\eta}{dz} = \frac{1}{c^2}(\zeta - \eta), \\ \tau \frac{d\zeta}{dz} = \frac{1}{c^2}(\phi - \zeta), \end{cases} \quad (15)$$

which is the slow system. When $\tau \rightarrow 0$, it can be verified that $\eta \rightarrow \phi$ from the latter two equations of system (15). At this limit status, it reduces to the unperturbed model

$$\begin{cases} \frac{d\phi}{dz} = \varphi, \\ \frac{d\varphi}{dz} = -\phi^2 - \phi^3, \end{cases} \quad (16)$$

which is a Hamiltonian system with the energy function

$$H(\phi, \varphi) = \frac{1}{2}\varphi^2 + \frac{1}{3}\phi^3 + \frac{1}{4}\phi^4. \quad (17)$$

By the bifurcation theory of dynamical systems, system (16) has two equilibria $E_1(0, 0)$, $E_2(-1, 0)$. By the Jacobian matrix of vector field of (16) at each equilibrium, one gets E_1 is saddle, E_2 is center. The corresponding energy function values are $H(0, 0) = 0$, $H(-1, 0) = -1/12$. Let $\Gamma_h := \{(\phi, \varphi) : H(\phi, \varphi) = h\}$

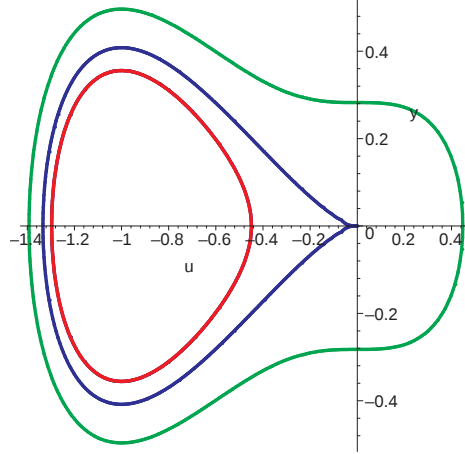


Fig. 1 Phase portraits of (16).

correspond to closed orbits of (16) for each $h \in (-1/12, 0)$. We give the phase portraits of (16) in Figures 1. Denote $\phi(z, c)$ the periodic wave solutions of system (13) and $\phi_0(z)$ the solitary wave solution which corresponds to the homoclinic orbit connecting to the origin. For the corresponding traveling wave system of delayed BBM equation (7), we obtain the following statements.

Theorem 1 *For any given traveling wave speed $c_1 \in ((\frac{7}{18})^{\frac{1}{3}}, \frac{1}{\sqrt[3]{2}})$, there exists $\tau_0(c) > 0$ such that when $0 < \tau < \tau_0(c)$, $c = c_0 + O(\tau)$, then Eq. (7) has a unique isolated periodic wave solution $u(x + ct)$, which given by $u(x + ct) = c\phi(c(x + ct))$, satisfying*

$$\lim_{\substack{(c, \tau) \rightarrow (\frac{1}{\sqrt[3]{2}}, 0), \\ 0 < \tau < \tau_0(c)}} \phi(x + ct) \rightarrow -1,$$

$$\lim_{\substack{(c, \tau) \rightarrow ((\frac{7}{18})^{\frac{1}{3}}, 0), \\ 0 < \tau < \tau_0(c)}} \phi(x + ct) \rightarrow \phi_0(x + ct).$$

We consider proving that the periodic wave persists for sufficiently small $\tau > 0$ by using geometric singular perturbation theory. Taking the time scaling

$z = \tau s$ into (15), it yields

$$\begin{cases} \frac{d\phi}{ds} = \tau\varphi, \\ \frac{d\varphi}{ds} = \tau \left(-\phi^2 - \eta\phi^2 - \frac{\tau}{c}\varphi \right), \\ \frac{d\eta}{ds} = \frac{1}{c^2}(\zeta - \eta), \\ \frac{d\zeta}{ds} = \frac{1}{c^2}(\phi - \zeta). \end{cases} \quad (18)$$

As introduced in Section 2, the time scale given by z is slow and s is fast, and the two systems are equivalent when $\tau \neq 0$. Meanwhile, system (18) is the fast system. The limit system of (18) with $\tau \rightarrow 0$ is given by

$$\begin{cases} \frac{d\phi}{ds} = 0, \\ \frac{d\varphi}{ds} = 0, \\ \frac{d\eta}{ds} = \frac{1}{c^2}(\zeta - \eta), \\ \frac{d\zeta}{ds} = \frac{1}{c^2}(\phi - \zeta), \end{cases} \quad (19)$$

which is the layer system. In system (15), when $\tau \rightarrow 0$, it yields the reduced system naturally

$$\begin{cases} \frac{d\phi}{dz} = \varphi, \\ \frac{d\varphi}{dz} = -\phi^2 - \phi^3, \\ \zeta - \eta = 0, \\ \phi - \zeta = 0. \end{cases} \quad (20)$$

Then the critical manifold is determined as

$$M_0 = \{(\phi, \varphi, \eta, \zeta) \in R^4 : \eta = \phi, \zeta = \phi, -1/3 < 2\varphi^2 + 4/3\phi^3 + \phi^4 < 0\}, \quad (21)$$

which is a four-dimensional submanifold in R^4 . Clearly, M_0 is a compact invariant manifold. The Jacobian matrix of the layer system (19) is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} & \frac{1}{c^2} \\ \frac{1}{c^2} & 0 & 0 & -\frac{1}{c^2} \end{pmatrix},$$

which always has four eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = -\frac{1}{c^2}$, that is, the Jacobian matrix has exactly two eigenvalues with nonzero real part at all its singularities. Then M_0 is normally hyperbolic. Notice that the set M_0 is given as the graph of the C^∞ . From Lemma 1, it deduces that for $\tau > 0$ is sufficiently

small, there exists a submanifold M_τ in R^4 , which is locally invariant under the flow of system (18), within the Hausdorff distance $O(\tau)$ of M_0 . From Lemma 2, M_τ can be expressed by the graph of a C^r function with $0 < r < +\infty$, which given as the form

$$M_\tau = \{(\phi, \varphi, \eta, \zeta) \in R^4 : \eta = \phi + g(\phi, \varphi, \tau), \zeta = \phi + h(\phi, \varphi, \tau)\},$$

where g, h are smooth functions and satisfy that $g(\phi, \varphi, 0) = 0, h(\phi, \varphi, 0) = 0$. Then the functions can be expanded into Taylor series

$$g(\phi, \varphi, \tau) = \tau g_1(\phi, \varphi) + O(\tau^2), \quad h(\phi, \varphi, \tau) = \tau h_1(\phi, \varphi) + O(\tau^2).$$

Then, it has

$$\frac{d\eta}{dz} = \varphi + O(\tau), \quad \frac{d\zeta}{dz} = \varphi + O(\tau). \quad (22)$$

Substituting (22) into the last two expressions in (15) and comparing the coefficients of τ , it obtains that $g_1 = -2c^2\varphi, h_1 = -c^2\varphi$. Therefore, the slow system (15) restricted on M_τ is reduced into a regular perturbed system

$$\begin{cases} \frac{d\phi}{dz} = \varphi, \\ \frac{d\varphi}{dz} = -\phi^2 - \phi^3 + \tau \left(2c^2\phi^2 - \frac{1}{c} \right) \varphi + O(\tau^2). \end{cases} \quad (23)$$

System (23) reduces to the non-delay model (16) when $\tau = 0$. Then system (23) is a near-Hamiltonian system. Since for $h \in (-\frac{1}{12}, 0)$, $H(\phi, \varphi) = h$ defines a family of periodic orbits Γ_h of system (23)| $_{\tau=0}$, where $H(\phi, \varphi)$ is the Hamiltonian function given by (17).

Denote $A(a(h), 0)$ the intersection point of Γ_h on the positive ϕ -axis, and T the period of Γ_h . For $\tau > 0$ sufficiently small, let $\Gamma_{h,\tau}$ be a piece of orbit starting from the point $A(a(h), 0)$ at time $z = 0$, and $B(b(h, \tau), 0)$ be the first intersection point of $\Gamma_{h,\tau}$ return to positive ϕ -axis at time $z = z^*(\tau)$ for h_τ is in a sufficiently small neighbourhood of h . By [33], the displacement function between $B(b(h, \tau), 0)$ and $A(a(h), 0)$ is given by

$$\begin{aligned} d(h, \tau, c) &= \int_{AB} dH = \int_{AB} \frac{\partial H}{\partial \phi} d\phi + \frac{\partial H}{\partial \varphi} d\varphi = \int_{AB} (\phi^2 + \phi^3) d\phi + \varphi d\varphi \\ &= \int_0^{z^*(\tau)} \left\{ (\phi^2 + \phi^3)\varphi + \left[-\phi^2 - \phi^3 + \tau \left(2c^2\phi^2 - \frac{1}{c} \right) \varphi + O(\tau^2) \right] \varphi \right\} dz \\ &= \tau \int_0^{z^*(\tau)} \left[\left(2c^2\phi^2 - \frac{1}{c} \right) + O(\tau) \right] \varphi dz \\ &= \tau F(h, \tau, c). \end{aligned}$$

By continuousness theorem, we have

$$\lim_{\tau \rightarrow 0} \Gamma_{h,\tau} = \Gamma_h, \quad \lim_{\tau \rightarrow 0} b(h, \tau) = a(h), \quad \lim_{\tau \rightarrow 0} z^*(\tau) = T(h).$$

Thus,

$$F(h, \tau, c) = M(h, c) + O(\tau),$$

where

$$M(h, c) = F(h, 0, c) = \oint_{\Gamma_h} \left(2c^2 \phi^2 - \frac{1}{c} \right) \varphi d\phi. \quad (24)$$

The function $M(h, c)$ is called Melnikov function and also called Abelian integrals in some references. In order to investigate the existence of periodic waves for the system (23), the zero of the displacement function $d(h, \tau)$ is necessary. According to (24) and the Poincaré bifurcation theory introduced in [34], it suffices to investigate the zero of the Melnikov function $M(h, c)$ in (24). Setting that

$$J_n(h) = \oint_{\Gamma_h} \phi^n \varphi d\phi, \quad (25)$$

where $n \in \mathbb{N}$. Then, Melnikov function can be expressed by

$$M(h, c) = 2c^2 J_2 - \frac{1}{c} J_0. \quad (26)$$

We have the following lemmas on the properties of $J_n(h)$.

Lemma 3 For $h \in (-\frac{1}{12}, 0)$, $J'_0(h) > 0$ and $J_0(h) > 0$.

Proof According to Eq. (17), we obtain $\frac{\partial \varphi}{\partial h} = \frac{1}{\varphi}$. Thus, it yields

$$J'_n(h) = \oint_{\Gamma_h} \phi^n \frac{\partial \varphi}{\partial h} d\phi = \oint_{\Gamma_h} \frac{\phi^n}{\varphi} d\phi,$$

Therefore, it has

$$J'_0(h) = \oint_{\Gamma_h} \frac{1}{\varphi} d\phi = \int_0^{T(h)} \frac{1}{\varphi} \varphi dz = \int_0^{T(h)} dz = T(h) > 0,$$

where $T(h)$ is the period of Γ_h . It implies that $J_0(h)$ is increasing strictly for $h \in (-\frac{1}{12}, 0)$. What's more, when $h \rightarrow -\frac{1}{12}$, Γ_h approaches to the center $(-1, 0)$, implying that $\varphi \rightarrow 0$. Thus, we have

$$J_0\left(-\frac{1}{12}\right) = \lim_{h \rightarrow -\frac{1}{12}} \oint_{\Gamma_h} \varphi d\psi = \lim_{\varphi \rightarrow 0} \int_0^{T(h)} \varphi^2 dz = 0.$$

Combining with $J'_0(h) > 0$ gives that $J_0(h) > 0$ for $h \in (-\frac{1}{12}, 0)$. The proof is finished.

From Lemma 3, the following ratio is well defined as $X(h) = \frac{J_2(h)}{J_0(h)}$. Combining with (26), we have

$$M(h, c) = 2c^2 J_0 \left(X(h) - \frac{1}{2c^3} \right). \quad (27)$$

The following proposition is needed for proving Theorem 2.1. In the rest of this section we shall give the proof.

Proposition 1 For $h \in (-\frac{1}{12}, 0)$, $X'(h) < 0$ and

$$\frac{7}{9} < X(h) < 1, \quad \lim_{h \rightarrow -\frac{1}{12}} X(h) = 1, \quad \lim_{h \rightarrow 0} X(h) = \frac{7}{9}.$$

In order to prove Proposition refprop1, some lemmas is meaningful to understand the property of $J_n(h)$ and $X(h)$.

Lemma 4

$$J_0(0) = -\frac{4\sqrt{2}\pi}{27}, \quad J_2(0) = -\frac{14\sqrt{2}\pi}{243}.$$

Then the ratio values at $h = 0$ is obtained

$$\frac{J_2(0)}{J_0(0)} = \frac{7}{9}$$

Proof As $h = 0$, it follows from (17) that $\varphi = \pm\sqrt{\phi^2(-\frac{2}{3} - \frac{\phi^2}{2})}$. Then by direct computation, we have

$$\begin{aligned} J_0(0) &= \oint_{\Gamma_h} \varphi d\phi = 2 \int_{-\frac{4}{3}}^0 \phi \sqrt{-\frac{2}{3} - \frac{\phi^2}{2}} d\phi = -\frac{4\sqrt{2}\pi}{27}, \\ J_2(0) &= \oint_{\Gamma_h} \phi^2 \varphi d\phi = 2 \int_{-\frac{4}{3}}^0 \phi^3 \sqrt{-\frac{2}{3} - \frac{\phi^2}{2}} d\phi = -\frac{14\sqrt{2}\pi}{243}. \end{aligned}$$

The proof is completed.

Lemma 5 At $h = -\frac{1}{12}$, the following ratio value holds:

$$\frac{J_2(-\frac{1}{12})}{J_0(-\frac{1}{12})} = \lim_{h \rightarrow -\frac{1}{12}} \frac{J_2(h)}{J_0(h)} = 1.$$

Proof Taking variable transformations $\phi = -1 + r \cos \theta$, $\varphi = r \sin \theta$ to move center to the origin, and denoting $\rho = \sqrt{h + \frac{1}{12}}$. From $H(\phi, \varphi) - h = 0$ it has

$$G(r, \rho, \theta) := \frac{1}{4}r^4 \cos^4 \theta + \frac{2}{3}r^3 \cos^3 \theta + \frac{1}{2}r^2 - \rho^2 = 0, \quad (28)$$

which can be rewritten as

$$\frac{r^2}{2} \left(1 + \frac{1}{2}r^2 \cos^4 \theta + \frac{4}{3}r \cos^3 \theta \right) = \rho^2. \quad (29)$$

Notice that $r \geq 0$ and $\rho \geq 0$, then Eq. (29) is equivalent to $\frac{r}{\sqrt{2}} \sqrt{1 + \frac{1}{2}r^2 \cos^4 \theta + \frac{4}{3}r \cos^3 \theta} = \rho$. Define that

$$\tilde{G}(r, \rho, \theta) := \frac{r}{\sqrt{2}} \sqrt{1 + \frac{1}{2}r^2 \cos^4 \theta + \frac{4}{3}r \cos^3 \theta} - \rho = 0. \quad (30)$$

When $h \rightarrow -\frac{1}{12}$, then $\rho \rightarrow 0$ and $r \rightarrow 0$. Therefore,

$$\widetilde{G}(0, 0, \theta) = 0, \text{ and } \frac{\partial \widetilde{G}(r, \rho, \theta)}{\partial r} \Big|_{(r, \rho)=(0,0)} = \frac{1}{\sqrt{2}} \neq 0.$$

By implicit function theorem, there exists a unique solution $r = r(\rho, \theta)$ such that $\widetilde{G}(r(\rho), \rho, \theta) = 0$ for (r, ρ) near $(0, 0)$ define an analytic function of ρ near 0. From $r(0, \theta) = 0$, it obtains that $r(\rho, \theta)$ can be expanded as a Taylor series. Combing with (30), it derives $r(\rho, \theta)$ as the form

$$\begin{aligned} r(\rho, \theta) = & \sqrt{2}\rho + \frac{4}{3}\cos^3\theta\rho^2 + \frac{\sqrt{2}}{18}\cos^4\theta(4\cos^2\theta - 9)\rho^3 \\ & + \cos^7\theta\left(\frac{256}{27}\cos^2\theta - 4\right)\rho^4 + O(\rho^5). \end{aligned} \quad (31)$$

By Green's formula, we get

$$J_n(h) = \oint_{\Gamma_h} \mu^n \nu d\mu = \iint_{\text{int}\Gamma_h} \mu^n d\mu d\nu = \int_0^{2\pi} d\theta \int_0^{r(\rho, \theta)} (-1 + r \cos \theta)^n r dr. \quad (32)$$

Noticing $\rho = \sqrt{h + \frac{1}{12}}$ and substituting (31) into (32), it yields

$$\begin{aligned} J_0(h) &= 2\pi \left(h + \frac{1}{12}\right) + \frac{31}{12}\pi \left(h + \frac{1}{12}\right)^2 + O\left(\left(h + \frac{1}{12}\right)^3\right), \\ J_2(h) &= 2\pi \left(h + \frac{1}{12}\right) - \frac{5}{12}\pi \left(h + \frac{1}{12}\right)^2 + O\left(\left(h + \frac{1}{12}\right)^3\right), \end{aligned}$$

for $0 < h + \frac{1}{12} \ll 1$. Therefore, it yields

$$\frac{J_2(-\frac{1}{12})}{J_0(-\frac{1}{12})} = \lim_{h \rightarrow -\frac{1}{12}} \frac{J_2(h)}{J_0(h)} = 1.$$

This is the completes proof.

Lemma 6 $J_n(h) = \sum_{i=0}^n (-1)^i C_n^i I_i(h)$, where $I_i(h) = \oint_{H^*=h} \mu^i \nu d\mu$, $\mu = 1 + \phi$, $\nu = \varphi$, and

$$H^*(\mu, \nu) = H(\mu - 1, \nu). \quad (33)$$

Particular, for $n = 0, 2$, it has

$$J_0(h) = I_0(h), \quad J_2(h) = I_2(h) - 2I_1(h) + I_0(h). \quad (34)$$

Proof On the transformation, it obtains

$$\begin{aligned} J_n(h) &= \oint_{\Gamma_h} \phi^n \varphi d\phi = \oint_{H^*=h} (\mu - 1)^n \nu d(\mu - 1) = \oint_{H^*=h} \left(\sum_{i=0}^n (-1)^i C_n^i \mu^i \nu \right) d\mu \\ &= \left(\sum_{i=0}^n (-1)^i C_n^i \right) \oint_{H^*=h} \mu^i \nu d\mu = \sum_{i=0}^n (-1)^i C_n^i I_i(h). \end{aligned}$$

Lemma 7 For $h \in (-\frac{1}{12}, 0)$, $\frac{J_2(h)}{J_0(h)}$ is strictly decreasing from 1 to $\frac{7}{9}$.

Proof Since $\frac{J_2(h)}{J_0(h)}$ is strictly monotonic on the interval $(-\frac{1}{12}, 0)$ is equivalent to any nontrivial linear combinations $\alpha_1 J_0(h) + \alpha_2 J_2(h)$ has at most one zero on $(-\frac{1}{12}, 0)$ counted with multiplicities, i.e., $\{J_0(h), J_2(h)\}$ are *ECT*-systems on $(-\frac{1}{12}, 0)$. Then the Chebyshev criteria [35, 36] is an effective tool to prove the monotonicity of the ratio function of two Abelian integrals. Taking the variable transformation $\mu = \phi + 1$, $\nu = \varphi$, then Eq. (33) changes into

$$H^*(\mu, \nu) = H(\mu - 1, \nu) = -\frac{1}{12} + \frac{1}{4}\mu^4 - \frac{2}{3}\mu^3 + \frac{1}{2}\mu^2. \quad (35)$$

Denoting $\tilde{h} = h + \frac{1}{12}$, then $H^*(\mu, \nu) = h$, $h \in (-\frac{1}{12}, 0)$ implies $\tilde{H}(\mu, \nu) = \tilde{h}$, $\tilde{h} \in (0, \frac{1}{12})$. Here

$$\tilde{H}(\mu, \nu) = H^*(\mu, \nu) + \frac{1}{12} = \Phi(\mu) + \frac{\nu}{2}, \quad (36)$$

with $\Phi(\mu) = \frac{1}{4}\mu^4 - \frac{2}{3}\mu^3 + \frac{1}{2}\mu^2$. Then for any given $\phi \in (-\frac{4}{3}, 0)$, it yields $\mu \in (-\frac{1}{3}, 1)$, there exists a unique analytic involution function $p(\mu) \in (-\frac{1}{3}, 1)$ such that

$$\Phi(\mu) = \Phi(p(\mu)), \quad p(\mu) \neq \mu. \quad (37)$$

From Lemma 6, setting $f_0(\mu) := 1$, $f_2(\mu) := \mu^2 - 2\mu + 1$. The discriminant function corresponding to $f_i(\mu)$ ($i=0,2$) is defined as

$$l_i(\mu) = \left(\frac{f_i}{\Phi'} \right) (\mu) - \left(\frac{f_i}{\Phi'} \right) (p(\mu)). \quad (38)$$

Factoring $\Phi(\mu) - \Phi(p)$, we obtain $\Phi(\mu) - \Phi(p) = \frac{1}{12}(\mu - p)q(\mu, p)$ with $q(\mu, p) = 3\mu^3 + 3\mu^2p + 3\mu p^2 + 3p^3 - 8\mu^2 - 8\mu p - 8p^2 + 6\mu + 6p$. From Eq. (37), $p(\mu)$ is an implicit function defined by the equation $q(\mu, p) = 0$. Thus, it has

$$\frac{d}{d\mu} l_i(\mu) = \frac{d}{d\mu} \left(\frac{f_i}{\Phi'} \right) (\mu) - \frac{d}{dp} \left[\left(\frac{f_i}{\Phi'} \right) (p(\mu)) \right] \frac{dp}{d\mu},$$

with

$$\frac{dp}{d\mu} = -\frac{\partial q(\mu, p)}{\partial \mu} / \frac{\partial q(\mu, p)}{\partial p}.$$

With aids of Maple, a direct computation possesses the following Wronskians

$$\begin{aligned} W[l_0(\mu)] &= -\frac{(\mu - p)w_1(\mu, p)}{\mu p(\mu - 1)^2(p - 1)^2}, \\ W[l_0(\mu), l_2(\mu)] &= -\frac{(\mu - p)^3 w_2(\mu, p)}{\mu^2 p^2 (\mu - 1)^3 (p - 1)^3 w_0(\mu, p)}, \end{aligned}$$

where

$$\begin{aligned} w_0(\mu, p) &= 3\mu^2 + 6\mu p + 9p^2 - 8\mu - 16p + 6, \\ w_1(\mu, p) &= \mu^2 + \mu p + p^2 - 2\mu - 2p + 1, \\ w_2(\mu, p) &= 18\mu^4 + 39\mu^3 p + 48\mu^2 p^2 + 39\mu p^3 + 18p^4 - 77\mu^3 - 145\mu^2 p - 145\mu p^2 \\ &\quad - 77p^3 + 119\mu^2 + 182\mu p + 119p^2 - 78\mu - 78p + 18. \end{aligned}$$

In order to find the zero of $w_i(\mu, p)$, a technical skill is calculating the resultant of $w_i(\mu, p)$ and $q(\mu, p)$ with respect to p . With the aids of mathematical software, it yields

$$\begin{aligned} \text{Res}(w_0, q, p) &= 144(3\mu^2 - 8\mu + 6)(3\mu + 1)^2(\mu - 1)^2, \\ \text{Res}(w_1, q, p) &= (-9\mu^4 + 18\mu^3 - 6\mu^2 - 6\mu - 1)(\mu - 1)^2, \\ \text{Res}(w_2, q, p) &= -864(81\mu^8 - 324\mu^7 + 414\mu^6 - 30\mu^5 - 236\mu^4 - 24\mu^3 \\ &\quad + 120\mu^2 + 54\mu + 9)(\mu - 1)^4. \end{aligned} \tag{39}$$

By Sturm's theorem, the polynomial $3\mu^2 - 8\mu + 6 = 0$ has no real root for $\mu \in (-\frac{1}{3}, 1)$. Thus, $w_0(\mu, p) = 0$ and $q(\mu, p) = 0$ have no common real root for $\mu \in (-\frac{1}{3}, 1)$, implying $W[l_0(\mu), l_2(\mu)]$ makes sense. Sturm's theorem is also used to verify the polynomial $-9\mu^4 + 18\mu^3 - 6\mu^2 - 6\mu - 1$ has no real zero for $\mu \in (-\frac{1}{3}, 1)$, then $W[l_0(\mu)] \neq 0$ for $\mu \in (-\frac{1}{3}, 1)$.

Similarly, the resultants $\text{Res}(w_2, q, p) \neq 0$ is used to verify that $W[l_0(\mu), l_2(\mu)] \neq 0$ for $\mu \in (0, 1)$, implying that $\{J_0(h), J_2(h)\}$ is an *ECT*-system on $(-\frac{1}{12}, 0)$. Thus, $\frac{J_2(h)}{J_0(h)}$ is strictly monotonic on $(-\frac{1}{12}, 0)$. Combining with Lemmas 4 and 5, the conclusion in this lemma is true.

According to Lemmas 4-6, the conclusion in Proposition 1 is obtained.

Now, we prove Theorem 1.

[Proof of Theorem 1.] By Eq. (27), we choose $c = c(h) = \left(\frac{X(h)}{2}\right)^{\frac{1}{3}}$ for each $h \in (-\frac{1}{12}, 0)$, then $M(h, c) = 0$ has a unique root, denoted by h^* . Moreover, $M'(h^*, c) \neq 0$, then h^* is simple root of $M(h, c) = 0$. It is not hard to give $c'(h) < 0$, since $X'(h) < 0$ for $h \in (-\frac{1}{12}, 0)$. Therefore, it holds

$$\left(\frac{7}{18}\right)^{\frac{1}{3}} < c(h) < \frac{1}{\sqrt[3]{2}}, \quad \lim_{h \rightarrow 0} c(h) = \left(\frac{7}{18}\right)^{\frac{1}{3}}, \quad \lim_{h \rightarrow -\frac{1}{12}} c(h) = \frac{1}{\sqrt[3]{2}}.$$

Applying the implicit function theorem, choosing $c = c(h) + O(\tau)$ such that $M(h, c) + O(\tau)$ has a unique zero near h^* , which implying that there exists sufficiently small $\tau_0(c) > 0$ such that when $0 < \tau < \tau_0(c)$, then $d(h, \tau, c)$ has a unique simple zero at $h = h^* + O(\tau)$. Therefore, system (15) has a unique isolated limit cycle near Γ_{h^*} , which decodes system (7) has a unique isolated periodic wave solution near $\phi(z, h^*)$. Furthermore, when $c \rightarrow \frac{1}{\sqrt[3]{2}}$ and $\tau \rightarrow 0$,

then $\Gamma_{h, \tau} \rightarrow (-1, 0)$ and $\phi(z, c, \tau, h) \rightarrow -1$. When $c \rightarrow \left(\frac{7}{18}\right)^{\frac{1}{3}}$ and $\tau \rightarrow 0$, then $\Gamma_{h, \tau} \rightarrow \Gamma_0$ and hence $\phi(z, c, \tau, h) \rightarrow \phi_0(z)$. These results lead to the conclusions of Theorem 1.

4 Numerical simulations

In this section, some numerical simulations are presented to verify the theoretical results obtained in the previous sections. It is shown that for $h \in (-\frac{1}{12}, 0)$, $X(h)$ is decreasing from 1 to $\frac{7}{9}$, we give the profile in Figure 2. Taking $h^* = -\frac{1}{13}$, $X(h^*) = 0.9901916789$, which in turns gives that $2c^3 = \frac{1}{X(h^*)} = 1.009905477$ leading to $M(h, c) = 0$. Consequently, setting $c = 0.7963125809$, $\tau = 0.01$, taking the initial value $(\phi, \varphi) = (-0.7, 0)$, it is not hard to find that the orbit is approaching to an isolated limit cycle, which is the unique persisted periodic orbit of the system (23), see Figure 3. Furthermore, fixing h, c and τ , taking the initial value $(\phi, \varphi) = (-0.8766925399, 0)$, we obtain the corresponding unique periodic waves solution of equation (7), see Figure 4. This is consistent to Theorem 1.

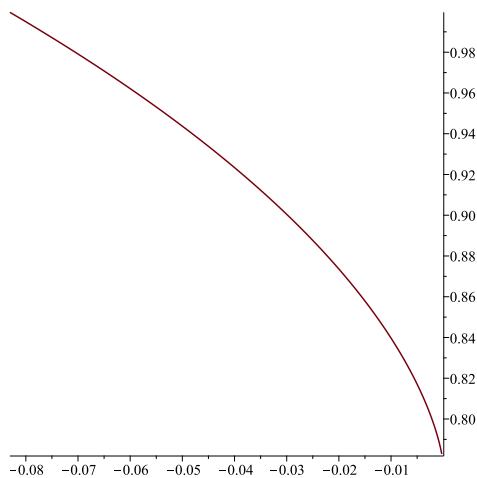


Fig. 2 The graph of $X(h)$ for $h \in (-\frac{1}{12}, 0)$.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

The authors declare that they have no conflict of interest.

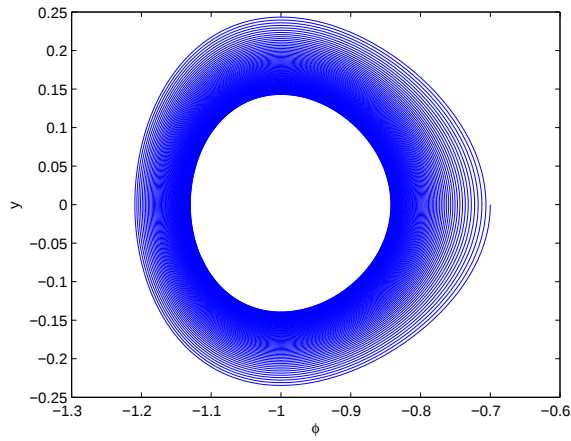


Fig. 3 The unique periodic orbit of the system (23).

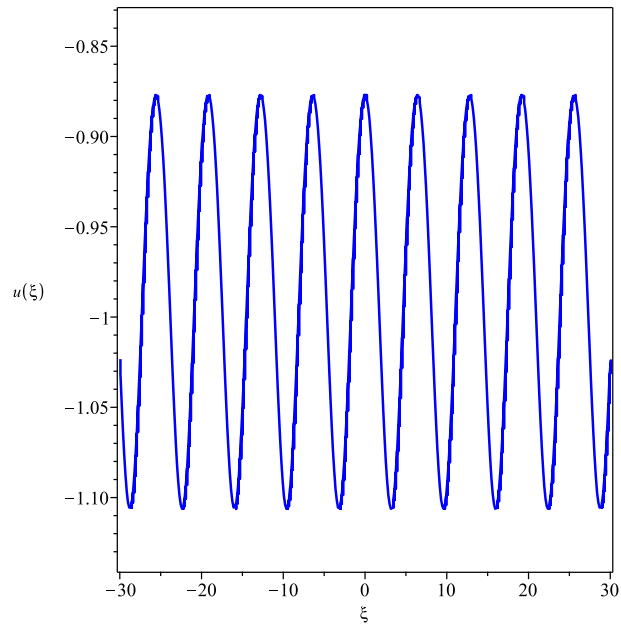


Fig. 4 The periodic wave of the delay equation (7).

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